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THE PERIODIC POINTS OF MORSE-SMALE ENDOMORPHISMS OF THE CIRCLE

BY

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ABSTRACT. Let $MS(S^1)$ denote the set of continuously differentiable maps of the circle with finite nonwandering set, which satisfy certain generic properties. For $f \in MS(S^1)$ let $P(f)$ denote the set of positive integers which occur as the period of some periodic point of f. It is shown that for $f \in MS(S^1)$ there are integers $m \geq 1$ and $n \geq 0$ such that $P(f) = \{m, 2m, 4m, \ldots, 2^n m\}$. Conversely, if m and n are integers, $m \geq 1$, $n \geq 0$, there is a map $f \in MS(S^1)$ with $P(f) = \{m, 2m, 4m, \ldots, 2^n m\}$.

1. Introduction. This paper is concerned with determining the possible orbit structures for a certain set of differentiable maps of the circle. For an introduction to the general theory of the orbit structures of differentiable maps of manifolds see [8] and [9]. Other papers on differentiable maps of the circle include [3], [4], [5], [6], and [7].

We let $MS(S^1)$ denote the set of continuously differentiable maps f of the circle to itself which satisfy the following properties (see §2 for definitions):

(1) $\Omega(f)$ (the nonwandering set) is finite.

(2) All periodic points of f are hyperbolic.

(3) No singularity of f is eventually periodic.

It can be shown that these conditions imply the following (see [2]):

(4) $\Omega(f)$ is the set of periodic points of f.

In this paper we ask the following question. Let $f \in MS(S^1)$. What are the possible periods of the periodic points of f?

More precisely, we let $P(f)$ denote the finite set of positive integers which are the periods of periodic points of f (so $n \in P(f)$ if and only if, for some $x \in S^1, f^n(x) = x$ and $f^k(x) \neq x \ \forall k < n$). We then ask what sets may occur as $P(f)$ for $f \in MS(S^1)$.

The following theorem which we proved in [2] gives a partial answer.

THEOREM A. *Let $f \in MS(S^1)$. There is a natural number $n(f)$ such that the period of any periodic point of f is $n(f)$ times a power of* 2.

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Theorem A implies for example that $f \in MS(S^1)$ cannot have a fixed point and a periodic point of period 3. However, Theorem A does not answer questions like the following: If $f \in MS(S^1)$ has a fixed point and a periodic point of period 4, must f have a periodic point of period 2?

The main results of this paper are the following two theorems which completely answer the question of what sets may occur as $P(f)$ for $f \in MS(S^1)$.

THEOREM B. *Let $f \in MS(S^1)$. There are integers m and n, $m \geqslant 1, n \geqslant 0$, such that $P(f) = \{m, 2m, 4m, \ldots, 2^n m\}$.*

THEOREM C. *Let m and n be integers, $m \geqslant 1, n \geqslant 0$. There is a map $f \in MS(S^1)$ with $P(f) = \{m, 2m, 4m, \ldots, 2^n m\}$.*

We close this section by remarking that properties (2) and (3) of the definition of $MS(S^1)$ are generic, i.e., true for a Baire subset of $C^1(S^1, S^1)$ (the space of continuously differentiable maps of the circle to itself with the C^1 topology). Also for $f \in MS(S^1)$ to be structurally stable we need two other technical conditions in addition to the ones given here (see [2]). However, $MS(S^1)$ as defined here has the property (easily checked) that if $f \in MS(S^1)$ then $f^n \in MS(S^1)$ for any positive integer n, and this property would not be true if we required $f \in MS(S^1)$ to satisfy the two additional conditions necessary for structural stability.

2. **Preliminary definitions and results.** Let $f \in C^1(S^1, S^1)$. A point $x \in S^1$ is said to be wandering if there is a neighborhood V of x with $f^n(V) \cap V = \varnothing$ for all positive integers n. The set of points on the circle which are not wandering is called the nonwandering set and denoted $\Omega(f)$.

A point $x \in S^1$ is called a singularity of f if $Df(x) = 0$ where $Df(x)$ denotes the derivative of f at x. x is called a periodic point of f if $f^n(x) = x$ for some positive integer n. x is said to be eventually periodic if $f^k(x)$ is a periodic point for some positive integer k.

Let $x \in S^1$ be a periodic point of f of period n. We say x is expanding if $|Df^n(x)| > 1$, and contracting if $|Df^n(x)| < 1$. We say x is hyperbolic if it is either expanding or contracting.

We will use the notation (a,b) to denote the open arc from a counterclockwise to b, and $[a, b]$ to denote the closed arc from a counterclockwise to b. We will say periodic points x and y are *adjacent* periodic points if there are no periodic points in one of the intervals (x,y), (y,x).

We state the following two lemmas which are proved in [2]. In Lemma 2, $\Omega_e(f)$ denotes the set of expanding periodic points of f, and $\Omega_c(f)$ denotes the set of contracting periodic points of f.

LEMMA 1. *Let $f \in MS(S^1)$. Suppose e and c are adjacent periodic points of f* *with e expanding and c contracting. If c is a fixed point of f then e is fixed by* f^2.

LEMMA 2. *Let* $f \in MS(S^1)$. *If f is onto, then cardinality* $\Omega_e(f) =$ *cardinality* $\Omega_c(f)$ *and the expanding and contracting periodic points alternate. If f is not onto, then cardinality* $\Omega_c(f) =$ *cardinality* $\Omega_e(f) + 1$.

In this case (if f has more than one periodic point) there is one pair of adjacent contracting periodic points, but otherwise the expanding and contracting periodic points alternate. Furthermore if x and y are the two adjacent contracting periodic points, with no periodic points in the open interval (x, y) *then* (x, y) *is not contained in* $f(S^1)$.

Let $f \in C^1(S^1, S^1)$. For a subset $A \subset S^1$, let orb (A) denote $\bigcup_{n>0} f^n(A)$.

Let e be an expanding fixed point of f. Let U be any open interval about e with $|Df(x)| > 1 \ \forall x \in U$. Let $W^u(e) =$ orb (U). Then $W^u(e)$ is clearly well defined. If e is orientation preserving (i.e. if $Df(e) > 0$), we let $W^u(e, cc)$ denote orb $([e, b])$ where b is any point with $|Df(x)| > 1 \ \forall x \in [e, b]$. Again $W^u(e, cc)$ is well defined. If e is orientation reversing we define $W^u(e, cc)$ by thinking of e as an orientation preserving fixed point of f^2.

Let c be a contracting fixed point of f. Let $W^s(c)$ (the stable manifold of c) denote the set of $x \in S^1$ such that c is a limit point of orb(x). We let *slsm*(c) denote the component of $W^s(c)$ which contains c. If c is a contracting periodic point of period n, we define $W^s(c)$ and *slsm*(c) by thinking of c as a fixed point of f^n.

The following two lemmas can be easily proved (see [2]). The proof of Lemma 3 uses Lemma 6 below.

LEMMA 3. *Let e be an expanding orientation preserving fixed point of* $f \in MS(S^1)$. *Let I denote the closure of* $W^u(e, cc)$. *Then I is a proper closed subinterval of* S^1 *and f has another fixed point in I (in addition to e).*

LEMMA 4. *Suppose c is a contracting fixed point of* $f \in MS(S^1)$, *and* $W^s(c) \neq S^1$. *If* e_1 *and* e_2 *are the endpoints of the open interval slsm(c), then one of the following must occur:*

(1) e_1 *and* e_2 *are expanding fixed points.*

(2) e_1 *and* e_2 *are expanding periodic points of period* 2.

(3) e_1 *is an expanding fixed point and* $f(e_2) = e_1$.

(4) $e_1 = e_2$ *is an expanding fixed point.*

The following lemma follows immediately by continuity (the intermediate value theorem).

LEMMA 5. *Suppose f is a continuous map of the circle to itself. Suppose K is a closed interval on* S^1 *with* $f(K) \supset K$ *and* $f(K) \neq S^1$. *Then f has a fixed point in K.*

We conclude this section with the following easy lemma which is one of the main consequences of the condition that $\Omega(f)$ is finite.

LEMMA 6. *Let $f \in MS(S^1)$ and let e be an expanding fixed point of f. If x is in the closure of $W^u(e)$ and $x \neq e$, then $f(x) \neq e$.*

PROOF. If $x \in \overline{W^u(e)}$ and $x \neq e$, and $f(x) = e$, then x is nonwandering but not periodic contradicting property (4) of $MS(S^1)$ (see §1). Here we have used property (3) to insure that for any neighborhood V of x, $f(V)$ contains a neighborhood of e. Q.E.D.

3. **Proof of Theorem B.** In following many of the proofs in this section it may be helpful for the reader to draw a circle and label important points on the circle in the correct order.

LEMMA 7. *Suppose $f \in MS(S^1)$ and f has a fixed point, and a periodic point of period $n \geqslant 4$, but no periodic points of period 2. Then f has an expanding fixed point adjacent to a contracting periodic point of period $k \geqslant 4$.*

PROOF. Clearly f must have adjacent periodic points p and q with p fixed and q of period at least 4. We have the following cases:

Case 1. Both p and q are contracting.

Case 2. Both p and q are expanding.

Case 3. p is contracting and q is expanding.

Case 4. q is contracting and p is expanding.

By Lemma 2, Case 2 is impossible, and by Lemma 1, Case 3 is impossible. In Case 4, the lemma is proved, so it suffices to look at Case 1.

Suppose p and q are contracting. Without loss of generality we may assume that there are no periodic points in (q,p). Then there must be adjacent periodic points a and b in the closed interval $[p,q]$ with a fixed and b of period at least 4, and $\{a,b\} \neq \{p,q\}$. As above we have the same four possible cases (with Case 2 and Case 3 impossible), but now by Lemma 2, a and b cannot both be contracting. Hence the only possibility is that a is expanding and b is contracting. Q.E.D.

LEMMA 8. *Let $f \in C^0(S^1, S^1)$. Let A and B be proper closed intervals of S^1 with $f(A) \supset B$ and $f(A) \neq S^1$. There is a closed interval $J \subset A$ with $f(J) = B$.*

PROOF. Let $B = [b_1, b_2]$. Since $f^{-1}(b_1) \cap A$ and $f^{-1}(b_2) \cap A$ are disjoint compact sets, there are points $v \in f^{-1}(b_1) \cap A$ and $w \in f^{-1}(b_2) \cap A$ such that if W is the open interval joining v and w with $W \subset A$ then $W \cap (f^{-1}(b_1) \cup f^{-1}(b_2)) = \varnothing$.

Let J be the closure of W. $f(J)$ must contain either B or the interval $[b_2, b_1]$. Since $f(A) \neq S^1$ but $f(A) \supset B$, $f(J)$ does not contain the interval $[b_2, b_1]$. Hence $f(J) \supset B$. Since no points in the interior of J are mapped to an endpoint of B we have $f(J) = B$. Q.E.D.

LEMMA 9. *Let* $f \in C^0(S^1, S^1)$. *Suppose* $I_1, \ldots, I_k$ *are proper closed intervals of* S^1 *with disjoint interiors, with the property that* $f(I_j) \supset I_{j+1}$ *for* $j = 1, \ldots, k-1$ *and* $f(I_k) \supset I_1$. *Suppose also that* $f(I_j) \neq S^1$ *for* $j = 1, \ldots, k$. *Finally, suppose that for all* $r \in \{1, \ldots, k-1\}$, *neither endpoint of* I_1 *is a periodic point of* f *of period* r. *Then* f *has a periodic point of period* k *in* I_1.

PROOF. By Lemma 8, there are closed intervals $J_1, \ldots, J_k$ such that $J_i \subset I_i$ for $i = 1, \ldots, k$ and $f(J_k) = I_1$ while $f(J_i) = J_{i+1}$ for $i = 1, \ldots, k-1$. It follows that $f^k(J_1) = I_1$. By Lemma 5, there is a fixed point p of f^k in $J_1 \subset I_1$.

If p is an endpoint of I_1, then, by hypothesis, p is a periodic point of f of period k. If p is an interior point of I_1, then since $f^i(p) \in J_{i+1} \subset I_{i+1}$ for $i = 1, \ldots, k-1$ and the intervals I_j have disjoint interior, p must be a periodic point of f of period k. Q.E.D.

LEMMA 10. *Suppose* $f \in MS(S^1)$ *and* f *has an expanding fixed point* e *adjacent to a periodic point* k_1 *of period* $n \geqslant 4$. *If* e *is orientation preserving then* f *has a periodic point of period* 2.

PROOF. Without loss of generality we may assume that there are no periodic points in (e, k_1) (the open arc from e counterclockwise to k_1).

Let $I = \overline{W^u(e, cc)}$. By Lemma 3, I is a proper interval and $k_1 \in I$. Since $f(I) = I$, we have orb $(k_1) \subset I$. Let orb $(k_1) = \{k_1, k_2, k_3, \ldots, k_n\}$ where we number these points in order counterclockwise around the circle beginning with k_1.

Consider the $n-1$ closed intervals, $[k_1, k_2], [k_2, k_3], \ldots, [k_{n-1}, k_n]$. Note that each of these intervals is contained in I, and $f(I) = I$. Hence for each $i = 1, \ldots, n-1$, $f([k_i, k_{i+1}])$ contains at least one interval $[k_j, k_{j+1}]$ with $j \neq i$. It follows (using Lemma 9) that f has a periodic point of period r_1 in I, with $2 \leqslant r_1 \leqslant n-1$.

Repeating the argument (of the last paragraph) $n-3$ times with the new periodic point in place of k_1, we see that f has a periodic point of period 2. Q.E.D.

LEMMA 11. *Let* $f \in MS(S^1)$. *Suppose* e_1 *and* e_2 *are distinct expanding fixed points of* f, *and* $\overline{W^u(e_1)} = S^1$ (*where* $\overline{W^u(e_1)}$ *denotes the closure of* $W^u(e_1)$). *Then* $\overline{W^u(e_2)} \neq S^1$.

PROOF. Since $\overline{W^u(e_1)} = S^1$, it follows from Lemma 6 that if $f(x) = e_1$ then $x = e_1$.

Let I be an open interval containing e_1 with $|Df(x)| > 1$ $\forall x \in I$, and the length of $f(I)$ less than the distance from e_1 to e_2. Let $J = S^1 - f(S^1 - I)$. Then J is an open interval containing e_1, because $f(S^1 - I)$ is compact and does not contain e_1.

Let $K = (a, b)$ be an open interval containing e_1 with the following properties:

(1) a and b are the same distance from e_1.
(2) $K \subset J \cap I$.
(3) e_2 is not in the closure of K. (This actually follows from (2).)

We claim $f(S^1 - K) \subset S^1 - K$. To prove this let $x \in S^1 - K$. First suppose $x \notin I$. Then $f(x) \in f(S^1 - I)$. Hence $f(x) \notin J$ so $f(x) \in S^1 - K$. Now suppose $x \in I$. Then the distance from $f(x)$ to e_1 must be greater than the distance from x to e_1. Since $x \in S^1 - K$, we have $f(x) \in S^1 - K$. This proves the claim.

Since e_2 is an interior point of $\overline{S^1 - K}$ and $f(S^1 - K) \subset S^1 - K$, it follows that $W^u(e_2) \subset S^1 - K$. Hence $\overline{W^u(e_2)} \neq S^1$. Q.E.D.

LEMMA 12. *Suppose $f \in MS(S^1)$ has a fixed point, and a periodic point of period at least* 4, *and no periodic points of period* 2. *Then f has an orientation reversing expanding fixed point e with $\overline{W^u(e)} \neq S^1$.*

PROOF. By Lemma 7, f must have an expanding fixed point e_0 adjacent to a contracting periodic point k of period at least 4. By Lemma 10, e_0 is orientation reversing. We may assume without loss of generality that there are no periodic points in (e_0, k). Also we may assume that $\overline{W^u(e_0)} = S^1$ (or else we are done).

We claim that f has an expanding fixed point $e_1 \neq e_0$. To prove this claim let $I = \overline{W^u(e_0, cc)}$. Then I is a proper closed interval and $f^2(I) \subset I$. It follows from Lemma 3 that f^2 has another fixed point $c \in I$ ($c \neq e_0$). Since f has no periodic points of period 2, c is a fixed point of f.

If c is expanding the claim is proven, so we may assume that c is contracting. By Lemma 4, and the fact that f has no periodic points of period 2, it follows that one endpoint e_2 of *slsm*(c) is an expanding fixed point. If $e_2 \neq e_0$ the claim is proven.

Suppose $e_2 = e_0$, i.e., e_0 is an endpoint of *slsm*(c). Let e_3 denote the other endpoint of *slsm*(c) (i.e., $e_3 \neq e_0$). Then since $\overline{W^u(e_0)} = S^1$, by Lemma 6 we have $f(e_3) \neq e_0$. Hence e_3 is an expanding fixed point by Lemma 4. This proves the claim that f has an expanding fixed point $e_1 \neq e_0$.

In $[e_0, e_1]$ there is a point k of period at least 4 and the fixed point e_1. Hence in the interval $[k, e_1]$ there must be adjacent periodic points c_1 and e with c_1 of period at least 4 and e fixed. Since $\overline{W^u(e_0)} = S^1$ implies that f is onto, it follows from Lemmas 1 and 2 that c_1 is contracting and e is expanding. By Lemma 10, e is orientation reversing. Since $\overline{W^u(e_0)} = S^1$ and $e \neq e_0$, by Lemma 11 we have $\overline{W^u(e)} \neq S^1$. Q.E.D.

THEOREM 13. *Let $f \in MS(S^1)$. Suppose f has a fixed point and a periodic point of period at least* 4. *Then f has a periodic point of period* 2.

PROOF. By Lemma 12, we may assume that f has an orientation reversing expanding fixed point e such that $\overline{W^u(e)} \neq S^1$. Let $\overline{W^u(e)} = [a, b]$. Note that

f maps the interval $[a, e]$ onto the interval $[e, b]$, and f maps the interval $[e, b]$ onto the interval $[a, e]$. This is true because f maps the interval $[a, b]$ onto itself, f is orientation reversing at e, and for $x \in [a, b]$, if $x \neq e$ then $f(x) \neq e$ by Lemma 6.

Pick $k_1 \in (e, b)$ so that for all $k \in (e, k_1)$, $f^2(k) \neq k$ and $(k, f^2(k)) \subset (e, b)$ (using the fact that e is an expanding fixed point). Let $[k_2, r]$ denote the image under f^2 of $[k_1, b]$. Then $k_2 \in (e, b)$. Let $k_0 \in (e, k_1) \cap (e, k_2)$ and let $I = [k_0, b]$. Then $f^2(I) \subset I$. Hence f^2 has a fixed point $y \in I$. But $f(I) \cap I = \varnothing$. Hence y is a periodic point of f of period 2. Q.E.D.

LEMMA 14. *Let $f \in MS(S^1)$. Suppose f has periodic points of periods k and m respectively with $k < m$. Then k divides m.*

PROOF. This follows immediately from Theorem A of §1 which is proved in [2]. Q.E.D.

THEOREM 15. *Let $f \in MS(S^1)$. Suppose f has a periodic point of period m and a periodic point of period $2^n m$ where $n > 1$. Then f has a periodic point of period $2m$.*

PROOF. Let $g = f^m$. Then $g \in MS(S^1)$ and g has a fixed point and a periodic point of period 2^n. By Theorem 13, g has a periodic point x of period 2. So $g^2(x) = x$ and $g(x) \neq x$. This implies $f^{2m}(x) = x$ and $f^m(x) \neq x$.

Let k be the period of x as a periodic point of f. Then k divides $2m$, but k does not divide m. By Lemma 14, $k = 2m$. Q.E.D.

We can now easily prove Theorem B. Recall that $P(f)$ denotes the set of positive integers which occur as the period of some periodic point of f.

THEOREM B. *Let $f \in MS(S^1)$. There are integers m and n, $m \geqslant 1$, $n \geqslant 0$, such that*

$$P(f) = \{m, 2m, 4m, \ldots, 2^n m\}.$$

PROOF. Let m be the smallest period of any periodic point of f. By Theorem A (see §1) the largest period of any periodic point of f is $2^n m$ for some nonnegative integer n. By Theorem A, $P(f) \subset \{m, 2m, 4m, \ldots, 2^n m\}$. But $m \in P(f)$ and $2^n m \in P(f)$ so by repeated application of Theorem 15 we have

$$P(f) = \{m, 2m, 4m, \ldots, 2^n m\}.$$

Q.E.D.

4. **Proof of Theorem C.** A major part of the proof of Theorem C is contained in the following lemma. We use the notation $\Omega_e(g)$ to denote the set of expanding periodic points of g.

LEMMA 16. *Let I be a proper closed interval of S^1. For any natural number $n \geqslant 2$, there is a continuously differentiable map g_n from I into itself with the following properties*:

(1) *g_n has exactly one singularity t in the interior of I.*

(2) *t is a periodic point of g_n of period 2^n.*

(3) *g_n has exactly one expanding fixed point e, and g_n is orientation reversing at e.*

(4) *$\Omega_e(g_n)$ consists of n periodic orbits of period* $1, 2, 4, \ldots, 2^{n-1}$ *respectively.*

(5) $\Omega(g_n) = orb\,(t) \cup \Omega_e(g_n)$.

(6) *The* (*one-sided*) *derivative of g_n is zero at the endpoints of I.*

PROOF. The proof is by induction. For $n = 2$, the map g_2 is constructed as in Figure 1. We use the notation t^k to denote $(g_2)^k(t)$ where k is a positive integer. In Figure 1, since g permutes the intervals $[t^1, t^3]$ and $[t^4, t^2]$, there are periodic points e_1 and e_2 of period 2 in these intervals. We can clearly arrange that $\Omega_e(g_2) = \{e, e_1, e_2\}$ and $\Omega(g_2) = \Omega_e(g_2) \cup$ orb (t), and that property (6) is satisfied.

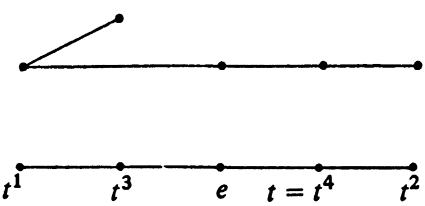

FIGURE 1
The map g_2

Proceeding now by induction we assume g_n is defined satisfying properties (1)–(6). We suppose we have drawn a diagram of g_n labeling the iterates of t, as we did with g_2 in Figure 1. We will modify this diagram of g_n and show that there is a map g_{n+1} corresponding to the modified diagram which satisfies properties (1)–(6).

Let $k_1, k_2, \ldots, k_{(2^n)}$ denote the powers of t $(1 \leqslant k_i \leqslant 2^n)$ on the diagram of g_n in order from left to right. For example, if $g_n = g_2$ then $k_1 = 1$, $k_2 = 3$, $k_3 = 4$, and $k_4 = 2$, since t^1, t^3, t^4, and t^2 are the iterates of t in order from left to right on the diagram of g_2. By induction we may assume that $|k_i - k_{i+1}| = 2^{n-1}$ for i odd.

Now in each of the intervals $[t^{(k_i)}, t^{(k_{i+1})}]$ where i is odd we add two points $t^{(l_i)}$ and $t^{(l_{i+1})}$, with $t^{(l_i)}$ to the left of $t^{(l_{i+1})}$, where $l_i = k_i + 2^n$ and $l_{i+1} = k_{i+1} + 2^n$. Then at the point $t^{(2^{n+1})}$ we write $t = t^{(2^{n+1})}$. Note that for the value of i with $k_i = 2^n$ we have $l_i = 2^n + 2^n = 2^{n+1}$.

It should be noted here that we are using the points $t^{(l_i)}$ and $t^{(k_i)}$ to describe the orbit of a new singularity t of g_{n+1}. A good way to think of what we have done is the following. Let $s = 2^n$. Relabel the points $t^1, t^2, \ldots, t^s$ as $p_1, p_2, \ldots, p_s$. For g_n, $t^s = t$ is its singularity. Near each p_i we pick a point q_i (on the appropriate side of p_i). We then modify the map g_n so that for the new map g_{n+1} the point q_s is a singularity and $g_{n+1}(p_s) = q_1$ while $g_{n+1}(q_s) = p_1$. Thus the successive iterates of q_s under g_{n+1} are $q_s, p_1, p_2, \ldots, p_s, q_1, q_2, \ldots, q_s, \ldots$. Finally we relabel $t = q_s$, $t^1 = p_1$, $t^2 = p_2$, etc. Then t is a singularity of g_{n+1} of period $2s = 2^{n+1}$.

This is the modified diagram we want. By construction and our induction hypothesis it follows that there is a map g_{n+1} with iterates of t as described in the modified diagram. (It may be helpful to see Figure 2 in which the diagram of g_3 is given, and compare with the diagram of g_2 in Figure 1.)

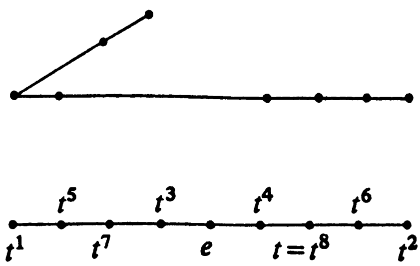

FIGURE 2
The map g_3

By construction, g_{n+1} satisfies property (2), and we can easily insure that properties (1), (3), and (6) are also satisfied. We must show g_{n+1} can be constructed to also satisfy properties (4) and (5).

Consider the 2^n intervals of the form $[t^{(k_i)}, t^{(l_i)}]$ with i odd or $[t^{(l_i)}, t^{(k_i)}]$ with i even, where $k_i = 1, \ldots, 2^n$. Since $l_i = k_i + 2^n$ it follows that these intervals are permuted by g_{n+1}. Hence we can arrange that in each of these intervals there is an expanding periodic point of period 2^n, with everything else in these intervals in the stable manifold of the orbit of t.

Let K be the union of the 2^{n-1} intervals of the form $[t^{(k_i)}, t^{(k_{i+1})}]$, i odd, $i = 1, 3, \ldots, 2^n - 1$. Then by construction, $g_{n+1}(K) = K$, and we can label the intervals $[t^{(k_i)}, t^{(k_{i+1})}]$ as $K_1, K_2, \ldots, K_{(2^{n-1})}$ where $g_{n+1}(K_i) = K_{i+1}$ for $i = 1, \ldots, 2^{n-1} - 1$ and $g_{n+1}(K_i) = K_1$ for $i = 2^{n-1}$.

Now consider the 2^{n-1} intervals of the form $[t^{(l_i)}, t^{(l_{i+1})}]$ where i is odd, $i = 1, 3, \ldots, 2^n - 1$. Let L denote the union of these intervals. Note that $[t^{(l_i)}, t^{(l_{i+1})}] \subset [t^{(k_i)}, t^{(k_{i+1})}]$ (for i odd) and

$$(g_{n+1})^{(2^{n-1})}([t^{(l_i)}, t^{(l_{i+1})}]) \supset [t^{(l_i)}, t^{(l_{i+1})}].$$

Hence we can arrange that in each interval $[t^{(i_i)}, t^{(i_{i+1})}]$ there is an expanding periodic point of period 2^{n-1}, and all other interior points of L are wandering.

So far we have shown that we can construct g_{n+1} on the set K so that $g_{n+1}(K) \subset K$ and $\Omega(g_{n+1}|K)$ consists of three periodic orbits, namely expanding periodic orbits of period 2^n and 2^{n-1}, and the orbit of the singularity t of period 2^{n+1}.

Let B be the closure of an interval in $I - K$. Then for each $k > 0$, $(g_n)^k(B) \supset B$ if and only if $(g_{n+1})^k(B) \supset B$. Hence we can arrange that on $I - K$, g_{n+1} (like g_n) has $n - 1$ expanding periodic orbits of period $2^{n-2}, 2^{n-3}, \ldots, 2, 1$, with everything else in $I - K$ wandering. Thus g_{n+1} satisfies properties (1)–(6). Q.E.D.

We state the following easy lemma which is a special case of the more general Ω-stability theorems of [1] or [5] (a similar theorem may be found in [7]). Note that f may have a singularity which is also a contracting periodic point and still satisfy the hypothesis of the lemma.

LEMMA 17. *Suppose $f \in C^1(S^1, S^1)$ and f satisfies the following*:

(1) *$\Omega(f)$ is a finite set of periodic points.*

(2) *All periodic points of f are hyperbolic.*

(3) *All singularities of f are in the stable manifolds of contracting periodic points.*

Then for g sufficiently close to f in $C^1(S^1, S^1)$, g will also satisfy conditions (1), (2), *and* (3). *Furthermore, for each positive integer k, there will be a one-to-one correspondence between the periodic points of f and g of period k, with expanding* (*respectively, contracting*) *periodic points of f corresponding to expanding* (*respectively, contracting*) *periodic points of g.*

For an interval J of S^1 we define $MS(J)$ to be the set of continuously differentiable maps of J into itself which satisfy the same properties as in the definition of $MS(S^1)$ in §1.

LEMMA 18. *Let J be a proper closed interval of S^1. For any natural number n, $\exists f_n \in MS(J)$ such that*:

(1) $P(f_n) = \{1, 2, 4, \ldots, 2^n\}$.

(2) *The endpoints of J are expanding fixed points of f_n.*

PROOF. Let $I = [a, b]$ be a proper subinterval of J. For $n \geqslant 2$, let g_n be defined on I as in Lemma 16. For $n = 1$, let g_1 be a continuously differentiable map of I onto itself with $\Omega(g_1) = \{a, b, e\}$ where e is an orientation reversing expanding fixed point of g_1, a and b are contracting periodic points of period two with $Dg_1(a) = Dg_1(b) = 0$, and a and b are the only singularities of g_1.

Let $J = [c, d]$. We extend g_n to J as follows. Let g_n map the interval $[c, a]$ onto the interval $[c, g_n(a)]$ so that:

(A) c is an expanding fixed point of g_n.

(B) For each $x \in (c,a)$, $g_n(x)$ is to the right of x in $[c,d]$.

(C) g_n is continuously differentiable on $[c,a]$.

(D) $Dg_n(a) = 0$ and $Dg_n(x) \neq 0$ if $x \in (c,a)$.

We define g_n similarly on $[b,d]$. Then g_n is a continuously differentiable map of J onto itself, $\Omega(g_n)$ is a finite set of periodic points, and g_n satisfies properties (1) and (2) of the conclusion of the lemma.

By construction, all periodic points of g_n are expanding except for one contracting periodic orbit which contains all singularities of g_n. We can easily perturb g_n to a map f_n (i.e. there is a map, f_n, arbitrarily close to g_n in $C^1(J,J)$) for which this periodic orbit is replaced by the orbit of a contracting periodic point k, with no singularities in orb(k). By Lemma 17, we can make the perturbation small enough to insure that $\Omega(f_n)$ consists of periodic points, the periodic points of f_n correspond to those of g_n, and all singularities of f_n are in the stable manifolds of points in the orbit of k. Thus $f_n \in MS(J)$ and f_n satisfies (1) and (2). Q.E.D.

THEOREM C. *Let m and n be integers, $m \geqslant 1, n \geqslant 0$. There is a map $f \in MS(S^1)$ with $P(f) = \{m, 2m, 4m, \ldots, 2^n m\}$.*

PROOF. For $n = 0$ the theorem is obvious so we may assume $n \geqslant 1$. Let $I_1, \ldots, I_m$ be disjoint proper closed intervals of S^1, numbered in order around the circle. For $k = 1, \ldots, m-1$, let f be any orientation preserving diffeomorphism from I_k onto I_{k+1}. Then define f on I_m by $f = f_n \circ f^{-(m-1)}$ where $f_n: I_1 \to I_1$ is the map defined in Lemma 18, and $f^{-(m-1)}$ makes sense as f has been defined on $I_1, I_2, \ldots, I_{m-1}$.

Let $I_k = [a_k, b_k]$. Then for $k = 1, \ldots, m-1, f(a_k) = a_{k+1}$ and $f(b_k) = b_{k+1}$ while $f(a_m) = a_1$ and $f(b_m) = b_1$. Also, each of the sets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ form an expanding periodic orbit. Hence we can extend f to a continuously differentiable map of the circle with a contracting periodic orbit of period m in $S^1 - (I_1 \cup \cdots \cup I_m)$ and all other points of $S^1 - (I_1 \cup \cdots \cup I_m)$ wandering.

Then $f \in MS(S^1)$ and $P(f) = \{m, 2m, 4m, \ldots, 2^n m\}$. Q.E.D.

Finally, we remark that it is clear from the construction of the map f in Theorem C, that f can be made to be C^∞.

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